Mathematics: Analysis & Approaches SL & HL

1 Page Formula Sheet – First Examinations 2021 – Updated Version 1.3

Prior Learning SL & HL

Area: Parallelogram	$A = bh$, b = base, h = height
Area: Triangle	$A = \frac{1}{2}(bh)$, b = base, h = height
Area: Trapezoid	$A = \frac{1}{2}(a+b)h$, a, b = parallel sides, h = height
Area: Circle	$A = \pi r^2$, r = radius
Circumference: Circle	$C = 2\pi r$, r = radius
Volume: Cuboid	$V = lwh$, l = length, w = width, h = height
Volume: Cylinder	$V = \pi r^2 h$, r = radius, h = height
Volume: Prism	$V = Ah$, A = cross-section area, h = height
Area: Cylinder curve	$A = 2\pi rh$, r = radius, h = height
Distance between two points (x_1, y_1), (x_2, y_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Coordinates of midpoint	$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, for endpoints $(x_1, y_1), (x_2, y_2)$

Topic 1: Number and algebra - SL & HL

The nth term of an arithmetic sequence	$u_n = u_1 + (n-1)d$
Sum of n terms of an arithmetic sequence	$S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$
The nth term of a geometric sequence	$u_n = u_1 r^{n-1}$
Sum of n terms of a finite geometric seq.	$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, r \neq 1$
Compound interest	$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$ FV is future value, PV is present value, n is the number of years, k is the number of compounding periods per year, $r\%$ is the nominal annual rate of interest
Exponents & logarithms	$a^x = b \Leftrightarrow x = \log_a b$, $a, b > 0, a \neq 1$
Exponents & logarithms	$\log_a xy = \log_a x + \log_a y$ $\log_a \frac{x}{y} = \log_a x - \log_a y$ $\log_a x^m = m \log_a x$ $\log_a x = \frac{\log_b x}{\log_b a}$
The sum of an infinite geometric sequence	$S_\infty = \frac{u_1}{1 - r}$, $\lvert r \rvert < 1$
Binomial Theorem for $n \in \mathbb{N}$, $(a+b)^n =$	$a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$
Binomial coefficient	$\binom{n}{r} = {}^n\mathrm{C}_r = \frac{n!}{r!(n-r)!}$

Topic 1: Number and algebra - HL only

Combinations; Permutations	${}^n\mathrm{C}_r = \frac{n!}{r!(n-r)!}$; ${}^n\mathrm{P}_r = \frac{n!}{(n-r)!}$
Extension of Binomial Theorem, $n \in \mathbb{Q}$	$(a+b)^n = a^n\left(1 + n\left(\frac{b}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2 + \ldots\right)$
Complex numbers	$z = a + bi$
Modulus-argument (polar) & Exponential (Euler) form	$z = r(\cos\theta + i\sin\theta) = re^{i\theta} = r\mathrm{cis}\theta$
De Moivre's theorem	$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta} = r^n \mathrm{cis} n\theta$

Topic 2: Functions – SL & HL

Equations of a straight line	$y = mx + c$; $ax + by + d = 0$; $y - y_1 = m(x - x_1)$
Gradient formula	$m = \frac{y_2 - y_1}{x_2 - x_1}$
Axis of symmetry of a quadratic function	$f(x) = ax^2 + bx + c \Rightarrow x = -\frac{b}{2a}$
Solutions of a quadratic equation in the form $ax^2 + bx + c = 0$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a \neq 0$
Discriminant	$\Delta = b^2 - 4ac$
Exponential and logarithmic functions	$a^x = e^{x \ln a}$; $\log_a a^x = x = a^{\log_a x}$ where $a, x > 0$, $a \neq 1$

Topic 2: Functions – HL only

Sum & product of the roots of polynomial equations of the form	$\sum_{r=0}^{n} a_r x^r = 0$ $\Rightarrow$ Sum is $\frac{-a_{n-1}}{a_n}$; product is $\frac{(-1)^n a_0}{a_n}$

Topic 3: Geometry and trigonometry – SL & HL

Distance between 2 points (x_1, y_1, z_1), (x_2, y_2, z_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
Coordinates of midpoint of a line with endpoints (x_1, y_1, z_1), (x_2, y_2, z_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
Volume: Right-pyramid	$V = \frac{1}{3}Ah$, A = base area, h = height
Volume: Right cone	$V = \frac{1}{3}\pi r^2 h$, r= radius, h = height
Area: Cone curve	$A = \pi rl$, r= radius, l = slant height
Volume: Sphere	$V = \frac{4}{3}\pi r^3$, r = radius
Surface area: Sphere	$A = 4\pi r^2$, r = radius
Sine rule	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
Cosine rule	$c^2 = a^2 + b^2 - 2ab\cos C$ $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$
Area: Triangle	$A = \frac{1}{2}ab\sin C$
Length of an arc	$l = r\theta$, r = radius, θ = angle in radians
Area of a sector	$A = \frac{1}{2}r^2\theta$, r = radius, θ = angle in radians
Identity for $\tan\theta$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
Pythagorean identity	$\cos^2\theta + \sin^2\theta = 1$
Double angle identities	$\sin 2\theta = 2\sin\theta\cos\theta$ $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $= 2\cos^2\theta - 1$ $= 1 - 2\sin^2\theta$

Topic 3: Geometry and trigonometry – HL only

Reciprocal trigonometric identities	$\sec\theta = \frac{1}{\cos\theta}$; $\mathrm{cosec}\,\theta = \frac{1}{\sin\theta}$
Pythagorean identities	$1 + \tan^2\theta = \sec^2\theta$; $1 + \cot^2\theta = \mathrm{cosec}^2\theta$
Compound angle identities	$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$ $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$ $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
Double angle identity for tan	$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$
Magnitude of a vector	$\lvert \boldsymbol{v} \rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$
Scalar product	$\boldsymbol{v} \cdot \boldsymbol{w} = v_1w_1 + v_2w_2 + v_3w_3$ $\boldsymbol{v} \cdot \boldsymbol{w} = \lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\cos\theta$ where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
Angle between two vectors	$\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert}$
Vector equ. of a line	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$
Parametric form of the equation of a line	$x = x_0 + \lambda l$, $y = y_0 + \lambda m$, $z = z_0 + \lambda n$
Cartesian equations of a line	$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$
Vector product	$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$ $\lvert\boldsymbol{v} \times \boldsymbol{w}\rvert = \lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\sin\theta$ where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
Area of a parallelogram	$A = \lvert\boldsymbol{v} \times \boldsymbol{w}\rvert$, where $\boldsymbol{v}$ and $\boldsymbol{w}$ form two adjacent sides of a parallelogram
Vector equ. of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
Equation of a plane	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ (using the normal vector)
Cartesian equ. of a plane	$ax + by + cz = d$

Topic 4: Statistics and probability - SL & HL

Interquartile range	$\mathrm{IQR} = Q_3 - Q_1$
Mean, $\bar{x}$, of a set of data	$\bar{x} = \frac{\sum_{i=1}^{k} f_i x_i}{n}$, where $n = \sum_{i=1}^{k} f_i$
Probability of an event A	$\mathrm{P}(A) = \frac{n(A)}{n(U)}$
Complementary events	$\mathrm{P}(A) + \mathrm{P}(A') = 1$
Combined events	$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
Mutually exclusive events	$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$
Conditional probability	$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$
Independent events	$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$
Expected value: Discrete random variable X	$\mathrm{E}(X) = \sum x\,\mathrm{P}(X = x)$
Binomial distribution Mean ; Variance	$X \sim \mathrm{B}(n, p)$ $\mathrm{E}(X) = np$; $\mathrm{Var}(X) = np(1 - p)$
Standardized normal variable	$z = \frac{x - \mu}{\sigma}$

Topic 4: Statistics and probability – HL only

Bayes' theorem	$\mathrm{P}(B \mid A) = \frac{\mathrm{P}(B)\mathrm{P}(A \mid B)}{\mathrm{P}(B)\mathrm{P}(A \mid B) + \mathrm{P}(B')\mathrm{P}(A \mid B')}$ $\mathrm{P}(B_i \mid A) = \frac{\mathrm{P}(B_i)\mathrm{P}(A \mid B_i)}{\mathrm{P}(B_1)\mathrm{P}(A \mid B_1) + \mathrm{P}(B_2)\mathrm{P}(A \mid B_2) + \mathrm{P}(B_3)\mathrm{P}(A \mid B_3)}$
Variance σ^2	$\sigma^2 = \frac{\sum_{i=1}^{k} f_i(x_i - \mu)^2}{n} = \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2$
Standard Deviation σ	$\sigma = \sqrt{\frac{\sum_{i=1}^{k} f_i(x_i - \mu)^2}{n}}$
Linear transformation of a single random variable	$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$
Expected value: Continuous random variable X	$\mathrm{E}(X) = \mu = \int_{-\infty}^{\infty} xf(x)\mathrm{d}x$
Variance	$\mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2] = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$
Variance of a discrete random variable X	$\mathrm{Var}(X) = \sum(x - \mu)^2\,\mathrm{P}(X = x)$ $= \sum x^2\,\mathrm{P}(X = x) - \mu^2$
Variance of a continuous random variable X	$\mathrm{Var}(X) = \int_{-\infty}^{\infty}(x - \mu)^2 f(x)\mathrm{d}x$ $= \int_{-\infty}^{\infty} x^2 f(x)\mathrm{d}x - \mu^2$

Topic 5: Calculus - SL & HL

Derivative of x^n	$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$
Integral of x^n	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$
Area between curve $y = f(x)$ & x-axis	$A = \int_a^b y\,dx$, where $f(x) > 0$
Derivative of $\sin x$	$f(x) = \sin x \Rightarrow f'(x) = \cos x$
Derivative of $\cos x$	$f(x) = \cos x \Rightarrow f'(x) = -\sin x$
Derivative of e^x	$f(x) = e^x \Rightarrow f'(x) = e^x$
Derivative of $\ln x$	$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$
Chain rule	$y = g(u)$, $u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
Product rule	$y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
Quotient rule	$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
Acceleration	$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$
Distance; Displacement travelled from t_1 to t_2	$\mathrm{dist} = \int_{t_1}^{t_2} \lvert v(t) \rvert\,dt$; $\mathrm{disp} = \int_{t_1}^{t_2} v(t)\,dt$
Standard integrals	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$ $\int \sin x\,dx = -\cos x + C$ $\int \cos x\,dx = \sin x + C$ $\int e^x\,dx = e^x + C$
Area enclosed by a curve and x-axis	$A = \int_a^b \lvert y \rvert\,dx$

Topic 5: Calculus – HL only

Derivative of $f(x)$ from first principles	$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$
Standard derivatives	$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$ $f(x) = \sec x \Rightarrow f'(x) = \sec x\tan x$ $f(x) = \mathrm{cosec}\,x \Rightarrow f'(x) = -\mathrm{cosec}\,x\cot x$ $f(x) = \cot x \Rightarrow f'(x) = -\mathrm{cosec}^2 x$ $f(x) = a^x \Rightarrow f'(x) = a^x(\ln a)$ $f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{x\ln a}$ $f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1 - x^2}}$ $f(x) = \arccos x \Rightarrow f'(x) = -\frac{1}{\sqrt{1 - x^2}}$ $f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1 + x^2}$
Standard integrals	$\int a^x\mathrm{d}x = \frac{1}{\ln a}a^x + C$ $\int \frac{1}{a^2 + x^2}\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$ $\int \frac{1}{\sqrt{a^2 - x^2}}\mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + C$, $\lvert x\rvert < a$
Integration by parts	$\int u\frac{\mathrm{d}v}{\mathrm{d}x}dx = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}dx$
Area enclosed by a curve and y-axis	$A = \int_a^b \lvert x \rvert\,dy$
Volume of revolution about x or y-axes	$V = \int_a^b \pi y^2\,dx$ or $V = \int_a^b \pi x^2\,dy$
Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$ where h is a constant (step length)
Integrating factor for $y' + P(x)y = Q(x)$	$e^{\int P(x)\mathrm{d}x}$
Maclaurin series	$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$
Maclaurin series for special functions	• $e^x = 1 + x + \frac{x^2}{2!} + \ldots$ • $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ • $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ • $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ • $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$